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Fractal Functions and Wavelet Expansions Based on Several Scaling Functions

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We present a method for constructing translation and dilation invariant functions spaces using fractal functions defined by a certain class of iterated function systems. These spaces generalize the C^0 function spaces constructed in [D. Hardin, B. Kessler, and P. R. Massopust, *J. Approx. Theory* **71** (1992), 104–120] including, for instance, arbitrarily smooth function spaces. These new function spaces are generated by several scaling functions and their integer-translates. We give necessary and sufficient conditions for these function spaces to form a multiresolution analysis of $L^2(\mathbb{R})$.

1. INTRODUCTION

Dilation and translation invariant function spaces play an important role in the theory of wavelet expansions. More precisely, let N be an integer greater than 1 and for $f: \mathbb{R} \to \mathbb{R}$ define the *dilation of f by N* to be

$$D_N f = f(\cdot/N), \tag{1.1}$$

and the *integer-translate of f* by

$$Tf = f(\cdot - 1). \tag{1.2}$$

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We are interested in linear function spaces that are invariant under D_N and T. Examples of such function spaces include the following:

(a) The spline space S_d^r with integer knots, i.e., all functions $f \in C^r(\mathbb{R})$ with $f|_{(n,n+1)}$ a polynomial of degree at most d. For $r = -1$ we let S_d^{-1} denote the piecewise polynomial functions of degree at most d with integer knots.

(b) The space V_0 generated by the integer translates of a single *scaling function* ϕ satisfying a dilation equation of the form

$$\phi(x/N) = \sum_{\ell \in \mathbb{Z}} p_\ell \phi(x - \ell). \tag{1.3}$$

Such spaces have been studied in [4, 5, 7, 14, 15, 16, 19, 20, 21, 22] and many others.

(c) The spaces $\mathscr{F}$ consisting of piecewise fractal interpolation functions as considered in [13]. The elements of $\mathscr{F}$ when restricted to $(n, n+1)$ satisfy "dilation" equations of the form

$$f(u_i(x-n)+n) = \lambda_{n,i}(x-n) + sf(x) \tag{1.4}$$

for $x \in (n, n+1)$ and $i = 0, \ldots, N-1$, where $u_i(x) = (x+i)/N$, and $\lambda_{n,i}(x)$ is an affine function of x and $|s| < 1$.

For each of the spaces S_{r+1}^r, $r = -1, 0, 1, 2, \ldots$, there exists a single function ϕ which together with its integer translates generates S_{r+1}^r. In the case $r = -1$, $\phi = \chi_{[0,1]}$. For $r \geq 0$, ϕ is the cardinal B-spline of degree $r+1$ (cf. [6] and references therein). Therefore, ϕ satisfies a dilation equation of the form (1.3) and so S_{r+1}^r is a special case of the spaces in (b). In general, however, the spaces S_d^r for $d > r+1$ are generated by the integer translates of a finite collection of (scaling) functions. The case S_d^{-1} was for example, investigated in [1].

The spaces of fractal interpolation functions studied in [13] also require—due to their construction—more than one scaling function. In Section 2, we extend the construction of translation- and dilation-invariant linear spaces by setting up a linear isomorphism between the space of real-valued functions bounded on compact subsets of $\mathbb{R}$ and a function space Λ whose elements are sequences $\boldsymbol{\lambda} = \{\lambda_{n,i}: n \in \mathbb{Z}, i = 0, \ldots, N-1\}$ of functions bounded on [0, 1]. The lift $\Delta_N: \Lambda \to \Lambda$ of D_N to this function space is given by

$$(\Delta_N \boldsymbol{\lambda})_{Nn+j,i} = \lambda_{n,j} \circ u_i + s(\lambda_{n,i} - \lambda_{n,j}), \tag{1.5}$$

where $n \in \mathbb{Z}$ and $i, j = 0, \ldots, N-1$. Equation (1.5) then leads to the construction of translation- and dilation-invariant spaces from Δ_N- and shift- (on n) invariant linear spaces of $\boldsymbol{\lambda}$'s. For instance, the linear subspace $\{\boldsymbol{\lambda} \in \Lambda\colon \lambda_{n,i}$ is a polynomial of degree $\leq d$ for all $n \in \mathbb{Z}$ and $i = 0, \ldots, N-1\}$ is clearly Δ_N- and shift-invariant. Similarly one may use spline spaces or even some $V_0|_{[0,1]}$ from (b). However, the functions constructed from such λ's are typically wildly discontinuous. In Theorem 2.3, we give necessary and sufficient conditions on $\boldsymbol{\lambda}$ and s such that the resulting function on $\mathbb{R}$ is C^r smooth. In particular, for $N = 2$ and each $s \in (-\frac{1}{2}, \frac{1}{2})$ we explicitly construct a translation and dilation invariant space of C^1 functions using quadratic $\lambda_{n,i}$'s and the conditions from Theorem 2.3. This space is generated by two compactly supported C^1 scaling functions. In the case $s = \frac{1}{8}$, the scaling functions are cubic Hermite interpolatory polynomials and, in the case $s = 0$, they are piecewise quadratic polynomials. Otherwise, the scaling functions are not piecewise polynomial and their derivatives are self-affine fractal functions. Thus, s is a free parameter that can be used to satisfy additional conditions. For instance, in Section 5, we find s so that the scaling functions and their integer-translates form an orthonormal set. Using these scaling functions, we construct in [11] compactly supported, continuous and orthonormal wavelets.

This approach can be generalized to scalar-valued functions on $\mathbb{R}^n$ using either n-simplices or n-cubes instead of intervals (cf. [12, 17]). The $n = 2$ case has been considered in [9]. The general case will be presented in forthcoming papers [10, 11, 12].

For a finite set $\{\phi^1, \phi^2, \ldots, \phi^A\}$ of scaling functions, the ϕ^i's satisfy a system of coupled dilation equations which can be written in the form of a single vector equation of the form (1.3). Here $\phi = (\phi^1, \ldots, \phi^A)^{\mathrm{T}}$ and the p_ℓ's are $A \times A$ matrices. In [11], we investigate solutions to these *matrix dilation equations*. In particular, we consider conditions on the matrix coefficients guaranteeing regularity of the solutions.

In Section 3, we focus on multiresolution analyses (MRAs) of $L^2(\mathbb{R})$ generated by several scaling functions. In this context, $\phi^1, \ldots, \phi^A$ generate an MRA of $L^2(\mathbb{R})$ if the following conditions hold:

Let $\{V_k\}_{k \in \mathbb{Z}}$ be a collection of closed subspaces of $L^2(\mathbb{R})$ satisfying

(a) Nestedness. $V_k \supset V_{k+1}, \qquad k \in \mathbb{Z}.$ (1.6)

(b) Separation. $\bigcup_{k \in \mathbb{Z}} V_k = \{0\}.$ (1.7)

(c) Density. $\overline{\bigcup V_k}^{L^2(\mathbb{R})} = L^2(\mathbb{R}).$ (1.8)

(d) $f \in V_k \leftrightarrow f(N\,\cdot\,) \in V_{k-1}, \qquad k \in \mathbb{Z}.$ (1.9)

(e) Let $\mathscr{B}_\Phi = \{\phi^\alpha(\cdot - \ell): \alpha = 1, \ldots, A;\ \ell \in \mathbb{Z}\}$. Then $\mathscr{B}_\Phi$ is a Riesz basis of V_0, i.e., $V_0 = \overline{\operatorname{span} \mathscr{B}_\Phi}^{L^2(\mathbb{R})}$ and there exist positive constants R_1 and R_2, called Riesz bounds, such that

$$R_1 \sum_{\alpha=1}^{A} \sum_{\ell \in \mathbb{Z}} |c_\ell^\alpha|^2 \le \left\| \sum_{\alpha=1}^{A} \sum_{\ell \in \mathbb{Z}} c_\ell^\alpha \phi^\alpha(\cdot - \ell) \right\|_2^2 \le R_2 \sum_{\alpha=1}^{A} \sum_{\ell \in \mathbb{Z}} |c_\ell^\alpha|^2, \quad (1.10)$$

for any square summable $\{c_\ell^\alpha\}$.

We give necessary and sufficient conditions for $\mathscr{B}_\Phi$ to be a Riesz basis of V_0. We also give sufficient conditions for the separation and density properties to hold. We verify that for each $s \in (-\frac{1}{2}, \frac{1}{2})$ the scaling functions for the C^1 example mentioned above generate an MRA of $L^2(\mathbb{R})$. We also construct associated *wavelets* that give an orthogonal direct sum decomposition of $L^2(\mathbb{R})$.

In Section 4, we give decomposition and reconstruction algorithms for these MRA's.

2. Translation- and Dilation-Invariant Subspaces

In this section we construct scaling functions generating multiresolution analyses. These functions are constructed using *fractal interpolation functions* which were introduced by Barnsley in [2]. Throughout this section $s \in (-1, 1)$ and N an integer greater than 1.

2.1. *Fractal Functions*

We first present an example to illustrate the construction of fractal interpolation functions. Given real numbers y_0, y_1, and y_2, and a parameter $s \in (-1, 1)$, we construct a continuous function on $[0, 1]$ interpolating $(0, y_0)$, $(\frac{1}{2}, y_1)$, and $(1, y_2)$ as the fixed point of the contractive (in the sup-norm) operator on bounded functions on $[0, 1]$,

$$(\Phi f)(x) = \lambda_i(u_i^{-1}(x)) + s f(u_i^{-1}(x)),$$

for all $x \in u_i([0, 1])$, $i = 0, 1$. Here $u_i(x) = (x + i)/2$ and $\lambda_i(x) = a_i x + b_i$ are chosen so that $(\Phi f)(0) = y_0$, $(\Phi f)(\frac{1}{2}) = y_1$, and $(\Phi f)(1) = y_2$, whenever $f(0) = y_0$, $f(\frac{1}{2}) = y_1$, and $f(1) = y_2$. Explicitly, we have $a_0 = (s-1)y_0 + y_1 - s y_2$, $a_1 = s y_0 - y_1 + (1-s) y_2$, $b_0 = (1-s) y_0$, and $b_1 = y_1 - s y_0$. The unique fixed point f^* is continuous and passes through the given interpolation points. The graph of f^* is *self-affine* in the following sense: Let $w_i(x, y) = (u_i(x), \lambda_i(x) + s y)$, $i = 0, 1$, and for any nonempty compact set $E \subseteq \mathbb{R}^2$ let $W(E) = w_0(E) \cup w_1(E)$. Then it is easy to verify that graph$(\Phi f) = W($graph$(f))$. So, graph(f) is a fixed point of W, and thus is a union of two smaller affine images of itself (see also [2, 3, 13 and 18]).

More generally, let $B(I)$ denote the Banach space of bounded real-valued functions on $I = [0, 1)$ with the sup-norm, and $\mathscr{B} = \otimes_{j=0}^{N-1} B(I)$ its N-fold direct product. Let $\lambda = (\lambda_0, \ldots, \lambda_{N-1}) \in \mathscr{B}$, and let u_i: $[0, 1) \to [0, 1)$ and v_i: $[0, 1) \times \mathbb{R} \to \mathbb{R}$ be as follows

$$u_i(x) = \frac{1}{N}(x + i), \tag{2.1a}$$

$$v_i(x, y) = \lambda_i(x) + sy, \tag{2.1b}$$

for $i = 0, \ldots, N-1$. Let $I_i = u_i(I) = [i/N, (i+1)/N)$ for $i = 0, \ldots, N-1$. Define Φ_λ: $B(I) \to B(I)$ by

$$\Phi_\lambda(f)(x) = v_i\big(u_i^{-1}(x), f\big(u_i^{-1}(x)\big)\big), \tag{2.2}$$

for $x \in I_i$ and $i = 0, 1, \ldots, N-1$.

Equations (2.2) and (2.1b) imply that Φ_λ is a contraction on $B(I)$ with contractivity $|s|$ and so Φ_λ has a unique fixed point $f_\lambda \in B(I)$. In the event that each λ_i is continuous and Φ_λ satisfies the join-up conditions

$$v_{i+1}\big(0, f_\lambda(0)\big) = v_i\big(1^-, f_\lambda(1^-)\big), \qquad i = 0, \ldots, N-1 \tag{2.3}$$

(note $f_\lambda(0) = \lambda_{N-1}(0)/(1-s)$ and $f_\lambda(1^-) = \lambda_0(1^-)/(1-s)$), then f_λ is continuous and, following [2], is called a *fractal interpolation function*. In general, we call f_λ a *fractal function* since the graph G of f_λ is typically a fractal set in $\mathbb{R}^2$. In fact, G is made up of smaller images of itself in the following sense: Let w_i: $[0, 1) \times \mathbb{R} \to [0, 1) \times \mathbb{R}$ be given by

$$w_i(x, y) = \big(u_i(x), v_i(x, y)\big) \tag{2.4}$$

for $i = 0, \ldots, N-1$, then 2.2 implies

$$G = \bigcup_{j=0}^{N-1} w_j(G) \tag{2.5}$$

That is, G is the attractor of the *iterated function system* $\{w_i\}_{i=0}^{N-1}$. We note that $w_i(G)$ satisfies a similar equation, namely

$$w_i(G) = \bigcup_{j=0}^{N-1} \big(w_i \circ w_j \circ w_i^{-1}\big)\big(w_i(G)\big), \tag{2.6}$$

for $i = 0, \ldots, N-1$.

Let ρ_i: $I_i \times \mathbb{R} \to I \times \mathbb{R}$ be the "rescaling function" defined by

$$\rho_i(x, y) = \big(u_i^{-1}(x), y\big).$$

Then, applying ρ_i to (2.6), we obtain

$$(\rho_i \circ w_i)(G) = \bigcup_{j=0}^{N-1} w_{i,j}((\rho_i \circ w_i)(G)),$$

where $w_{i,j} = (\rho_i \circ w_i) \circ w_j \circ (\rho_i \circ w_i)^{-1}$. A simple calculation shows that $w_{i,j} = (u_j, v_{i,j})$ where

$$\left.\begin{aligned} v_{i,j}(x,y) &= \big(\lambda_i \circ u_j + s(\lambda_j - \lambda_i)\big)(x) + sy \\ &=: \lambda_{i,j}(x) + sy \end{aligned}\right\} \tag{2.7}$$

Define $\lambda(i) = (\lambda_{i,0}, \ldots, \lambda_{i,N-1}) \in \mathscr{B}$ and observe that

$$\operatorname{graph}(f_{\lambda(i)}) = (\rho_i \circ w_i)(G) = \operatorname{graph}(f_\lambda \circ u_i), \tag{2.8}$$

where the second equality follows from f_λ being the fixed point of (2.2).

Let $\delta\colon \mathscr{B} \to \bigotimes_{j=0}^{N-1} \mathscr{B}$ be given by

$$\delta(\lambda) = (\lambda(0), \ldots, \lambda(N-1)). \tag{2.9}$$

In Section 2.2 we will identify elements in $\bigotimes_{\mathbb{Z}} \mathscr{B}$ with real-valued functions on $\mathbb{R}$. The dilation operator D_N as defined in (1.1) will be shown to correspond to the operator $\Delta_N\colon \bigotimes_{\mathbb{Z}} \mathscr{B} \to \bigotimes_{\mathbb{Z}} \mathscr{B}$ given by

$$(\Delta_N \boldsymbol{\lambda})_{Nn+j} = (\delta(\lambda_n))_j \tag{2.10}$$

for all $n \in \mathbb{Z}$ and $j \in \{0, \ldots, N-1\}$. Equation (2.10) can be written as Eq. (1.5) using (2.7) and (2.9). The following result gives the basic correspondence between elements in $\mathscr{B}$ and functions in $B(I)$.

THEOREM 2.1. *The mapping* $\lambda \overset{\theta}{\mapsto} f_\lambda$ *is a linear isomorphism from* $\mathscr{B}$ *to* $B(I)$.

Proof. From (2.1) and (2.2) it is clear that $\alpha f_\lambda + f_{\lambda'}$ is a fixed point of $\Phi_{\alpha\lambda+\lambda'}$, for all $\alpha \in \mathbb{R}$ and $\lambda, \lambda' \in \mathscr{B}$. By the uniqueness of the fixed point of $\Phi_{\alpha\lambda+\lambda'}$, we have $\alpha f_\lambda + f_{\lambda'} = f_{\alpha\lambda+\lambda'}$, and thus linearity. It is easy to see that $f_\lambda = 0$ iff $\lambda = 0$. To show surjectivity, let

$$\lambda_i(f) = f \circ u_i - sf,$$

for $i = 0, \ldots, N-1$. Then $\lambda(f) \in \mathscr{B}$ whenever $f \in B(I)$. Also, $f_{\lambda(f)} = f$. ∎

2.2. *The Correspondence between* D_N *and* Δ_N

Let $B_c(\mathbb{R})$ denote the Banach space of real-valued functions bounded on any compact subset of $\mathbb{R}$ with the sup-norm. An element $\mathbf{f} = \{f_n\}_{n=-\infty}^{+\infty} \in$

$\otimes_{\mathbb{Z}} B(I)$ is identified with an $f \in B_c(\mathbb{R})$ via the linear isomorphism

$$\mathbf{f} \overset{\tau}{\mapsto} \sum_{n \in \mathbb{Z}} f_n(\cdot - n)\chi_{[n,n+1)}, \tag{2.11}$$

where χ_A denotes the characteristic function of $A \subseteq \mathbb{R}$.

Note that $\theta\colon \mathscr{B} \to B(I)$ from Theorem 2.1 induces a linear isomorphism $\boldsymbol{\theta}\colon \otimes_{\mathbb{Z}} \mathscr{B} \to \otimes_{\mathbb{Z}} B(I)$. The next result shows the correspondence between $D_N\colon B_c(\mathbb{R}) \to B_c(\mathbb{R})$ defined in (1.2) and $\Delta_N\colon \otimes_{\mathbb{Z}} \mathscr{B} \to \otimes_{\mathbb{Z}} \mathscr{B}$ defined in (2.10).

THEOREM 2.2. *The following diagram commutes*:

$$\begin{array}{ccc} \bigotimes_{\mathbb{Z}} \mathscr{B} & \xrightarrow{\ \Delta_N\ } & \bigotimes_{\mathbb{Z}} \mathscr{B} \\ \downarrow{\scriptstyle \boldsymbol{\theta}} & & \downarrow{\scriptstyle \boldsymbol{\theta}} \\ \bigotimes_{\mathbb{Z}} B(I) & & \bigotimes_{\mathbb{Z}} B(I) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ B_c(\mathbb{R}) & \xrightarrow{\ D_N\ } & B_c(\mathbb{R}). \end{array} \tag{2.12}$$

Proof. Let $\mathbf{f} = \{f_n\}_{n \in \mathbb{Z}} \in \otimes_{\mathbb{Z}} B(I)$. Then

$$\begin{aligned} (D_N \circ \tau)(\mathbf{f})(x) &= \sum f_n\left(\frac{x - nN}{N}\right)\chi_{[n,n+1)}\left(\frac{x}{N}\right) \\ &= \sum_{i \in \mathbb{Z}} \sum_{j=0}^{N-1} f_i\left(\frac{x - iN}{N}\right)\chi_{[iN+j,iN+j+1)}(x). \end{aligned}$$

Thus

$$\left(\tau^{-1} \circ D_N \circ \tau(\mathbf{f})\right)_{iN+j}(x) = f_i\left(\frac{x+j}{N}\right) = f_i \circ u_j(x), \tag{2.13}$$

for $x \in I$. Equations (2.8) and (2.10) then give

$$\Delta_N = (\tau \circ \boldsymbol{\theta})^{-1} \circ D_N \circ (\tau \circ \boldsymbol{\theta}). \qquad \blacksquare$$

The other fundamental operator on $B_c(\mathbb{R})$ is the translation operator T as defined in (1.2). It is easy to see that the lift of T is the right-shift operator $\sigma\colon \otimes_{\mathbb{Z}} \mathscr{B} \to \otimes_{\mathbb{Z}} \mathscr{B}$ given by

$$\{\lambda_n\}_{n \in \mathbb{Z}} \mapsto \{\lambda_{n-1}\}_{n \in \mathbb{Z}}, \tag{2.14}$$

that is,

$$\sigma = (\tau \circ \boldsymbol{\theta})^{-1} \circ T \circ (\tau \circ \boldsymbol{\theta}). \tag{2.15}$$

Let r be a nonnegative integer. Later we will need the following characterization of $(\tau \circ \boldsymbol{\theta})^{-1} C^r(\mathbb{R}) \subseteq \mathscr{B}$.

Let $f \in C^r(\mathbb{R})$ and let $\boldsymbol{\lambda} = (\tau \circ \boldsymbol{\theta})^{-1}(f)$. Since the restriction of f to $I = [0,1)$ is the unique fixed point of $\Phi_{\boldsymbol{\lambda}_0}$ we have

$$f(u_i(x)) = \lambda_{0,i}(x) + sf(x), \tag{2.16}$$

for $x \in I$ and $i = 0, 1, \ldots, N-1$. Let $\tilde{C}^r(I)$ consist of the restrictions of functions $C^r(\bar{I})$ to I. Since $f \in \tilde{C}^r(I)$ it follows that $\boldsymbol{\lambda}_0 \in \bigotimes_{j=0}^{N-1} \tilde{C}^r(I) =: \mathscr{B}^r$.

Differentiating (2.16) m-times, $m = 0, 1, \ldots, r$, yields

$$f^{(m)}(u_i(x)) = N^m \lambda_{0,i}^{(m)}(x) + sN^m f^{(m)}(x), \tag{2.17}$$

for all $x \in I$ and $i = 0, 1, \ldots, N-1$. Using the continuity of $f^{(m)}$ at $u_{i-1}(1^-) = u_i(0) = i/N$, we obtain

$$\begin{aligned} L_i^m \boldsymbol{\lambda} := {} & (1 - sN^m)\big(\lambda_{0,i}^{(m)}(0) - \lambda_{0,i-1}^{(m)}(1)\big) \\ & + (sN^m)\big(\lambda_{0,0}^{(m)}(0) - \lambda_{0,N-1}^{(m)}(1)\big) = 0, \end{aligned} \tag{2.18}$$

for $m = 0, 1, \ldots, r$ and $i = 1, \ldots, N-1$, where L_i^m: $\bigotimes_{\mathbb{Z}} \mathscr{B}^r \to \mathbb{R}$. Similarly, using the continuity of $f^{(m)}$ at 0, we get

$$\hat{L}^m \boldsymbol{\lambda} := \lambda_{0,0}^{(m)}(0) - \lambda_{-1,N-1}^{(m)}(1^-) = 0, \tag{2.19}$$

for $m = 0, 1, \ldots, r$, where $\hat{L}^m$: $\bigotimes_{\mathbb{Z}} \mathscr{B}^r \to \mathbb{R}$.

Since $T^n f \in C^r(\mathbb{R})$ for all $n \in \mathbb{Z}$ we obtain

$$\sigma^n \boldsymbol{\lambda} \in \bigcap_{m=0}^{r} \left(\left(\bigcap_{i=1}^{N-1} \operatorname{Ker}(L_i^m) \right) \cap \operatorname{Ker}(\hat{L}^m) \right). \tag{2.20}$$

Let $\mathscr{C}^r$ be the set of all $\boldsymbol{\lambda} \in \bigotimes_{\mathbb{Z}} \mathscr{B}^r$ satisfying (2.20) for all $n \in \mathbb{Z}$. It follows from the above that $(\tau \circ \boldsymbol{\theta})^{-1} C^r(\mathbb{R}) \subseteq \mathscr{C}^r$. The next result gives us the reverse set containment.

THEOREM 2.3. *Suppose that* $|s| N^r < 1$. *Then* $(\tau \circ \boldsymbol{\theta})^{-1} C^r(\mathbb{R}) = \mathscr{C}^r$.

Proof. We explicitly denote the dependence of f_λ, Φ_λ, and $\mathscr{C}^r$ on s by $f_{\lambda,s}$, $\Phi_{\lambda,s}$, and $\mathscr{C}_s^r$, respectively. The crux of the proof lies in the following result.

LEMMA 2.1. *Let* $\lambda \in \mathscr{B}^1$ *satisfy*

$$(1 - sN^m)\big(\lambda_{i+1}^{(m)}(0) - \lambda_i^{(m)}(1)\big) + (sN^m)\big(\lambda_0^{(m)}(0) - \lambda_{N-1}^{(m)}(1)\big) = 0, \tag{2.21}$$

for $m = 0, 1$, *and* $i = 1, \dots, N-1$. *Suppose* $|s|N < 1$. *Then* $f_{\lambda, s} \in C^1(I)$ *and*

$$f_{N\lambda', Ns} = f'_{\lambda, s}. \tag{2.22}$$

Proof. It follows from (2.21) with $m = 0, 1$ and from $\lambda \in \mathscr{B}^1$ that $\Phi_{\lambda, s}$ is a contraction on $C^1(I)$ with contractivity $|s|N < 1$ in the C^1 topology. Thus $f_{\lambda, s} \in C^1(I)$. Comparing (2.17) and (2.16) we see that $f'_{\lambda, s}$ is a fixed point of $\Phi_{N\lambda', Ns}$. By the uniqueness of the fixed point the result follows. ∎

Now let $\boldsymbol{\lambda} \in \mathscr{C}_s^1$ and $f_{\boldsymbol{\lambda}, s} = (\tau_s \circ \boldsymbol{\theta})^{-1}\boldsymbol{\lambda}$. Since $\lambda_n, n \in \mathbb{Z}$, satisfies the hypotheses of Lemma 2.1, $f_{\boldsymbol{\lambda}, s}|_{[n, n+1)} \in C_s^1([n, n+1))$. Observe that

$$f'_{\boldsymbol{\lambda}, s}(n^+) = N\lambda'_{n,0}(0)/(1 - sN)$$

and

$$f'_{\boldsymbol{\lambda}, s}(n^-) = N\lambda'_{n-1, N-1}(1^-)/(1 - sN).$$

Thus (2.19) implies $f_{\boldsymbol{\lambda}, s} \in C^1(\mathbb{R})$ and (2.22) implies

$$f'_{\boldsymbol{\lambda}, s} = f_{N\boldsymbol{\lambda}', Ns}. \tag{2.23}$$

Note that $N\boldsymbol{\lambda}' \in \mathscr{C}_{Ns}^k$ whenever $\boldsymbol{\lambda} \in \mathscr{C}_s^{k+1}$ and the result follows by induction on r. ∎

Remark. This theorem implies that $\Delta_N \mathscr{C}^r \subseteq \mathscr{C}^r$.

2.3. *An Example*

Let π_n be the set of all polynomials of degree less than or equal to n whose domain is I, and let $\Pi_n = \otimes_{\mathbb{Z}}(\otimes_{j=0}^{N-1}\pi_n)$. Since $\lambda_i \circ u_j + s(\lambda_j - \lambda_i) \in \pi_n$ whenever $\lambda_i, \lambda_j \in \pi_n$, it follows from (2.10) that $\Delta_N \Pi_n \subseteq \Pi_n$. Let $\Pi_n^r = \Pi_n \cap \mathscr{C}^r$, i.e., $\boldsymbol{\lambda} \in \Pi_n^r$ if and only if $\boldsymbol{\lambda} \in \Pi_n^r$ and satisfies (2.20). Theorem 3.2 then implies that $\Delta_N \Pi_n^r \subseteq \Pi_n^r$.

EXAMPLE 1. In particular, we consider the case $N = 2$, $n = 2$, $r = 1$, and $|s| < \frac{1}{2}$. We note that the restrictions of these function to a single interval are integrals of self-affine fractal functions as considered in [3]. We identify π_2 with $\mathbb{R}^3$ via the correspondence $a + bx + cx^2 \mapsto (a, b, c)$. Hence $\lambda \in \pi_2 \otimes \pi_2$ can be identified with $v = (a_0, b_0, c_0, a_1, b_1, c_1) \in \mathbb{R}^6$

where $\lambda_i(x) = a_i + b_i x + c_i x^2$, $i = 0, 1$, and $\boldsymbol{\lambda} \in \Pi_2$ with $\{v_\ell = (a_{\ell,0}, b_{\ell,0}, c_{\ell,0}, a_{\ell,1}, b_{\ell,1}, c_{\ell,1})\}_{\ell \in \mathbb{Z}}$.

A function $g = \theta(\lambda)$ is in $C^1(I)$ if and only if λ satisfies (2.18) for $m = 0$ and 1. These equations are equivalent to

$$n_m \cdot v = 0, \qquad m = 0, 1, \tag{2.24}$$

where $n_0 = (2s - 1, s - 1, s - 1, 1 - 2s, -s, -s)$ and $n_1 = (0, 4s - 1, 4s - 2, 0, 1 - 4s, -4s)$. Thus $\theta(\pi_2 \otimes \pi_2) \cap C^1(I)$ is a four-dimensional linear space and each element g is uniquely determined by $g(0)$, $g'(0)$, $g(1)$, and $g'(1)$. Using (2.17) and (2.24), we cán solve for the corresponding λ in terms of these four values. In particular, let g_1, g_2, g_3, and g_4 be the basis elements

$$g_1(0) = g_2'(0) = g_3(1) = g_4'(1) = 1$$

and

$$g_1'(0) = g_1(1) = g_1'(1) = g_2(0) = \cdots = g_4(1) = 0.$$

Then the corresponding λ's are given by

$$\begin{aligned}
\lambda(1) &= \theta^{-1}(g_1) = \left(1 - s, 0, s - \tfrac{1}{2}, \tfrac{1}{2} - s, 2s - 1, \tfrac{1}{2} - s\right),\\
\lambda(2) &= \left(0, \tfrac{1}{2} - s, s - \tfrac{3}{8}, \tfrac{1}{8}, -\tfrac{1}{4}, \tfrac{1}{8}\right),\\
\lambda(3) &= \left(0, 0, \tfrac{1}{2} - s, \tfrac{1}{2}, 1 - 2s, s - \tfrac{1}{2}\right),\\
\lambda(4) &= \left(0, 0, s - \tfrac{3}{8}, s - \tfrac{1}{4}, -s, \tfrac{1}{4}\right).
\end{aligned}$$

Let $V_0 = (\tau \circ \boldsymbol{\theta})(\Pi_2^1)$. Then $f \in V_0$ is uniquely specified by $\{(f(\ell), f'(\ell))\}_{\ell \in \mathbb{Z}}$. Thus, if $\phi^1, \phi^2 \in V_0$ are such that $\operatorname{supp} \phi^1, \operatorname{supp} \phi^2 \subseteq [-1, 1]$, $\phi^1(0) = \phi^{2\prime}(0) = 1$ and $\phi^{1\prime}(0) = \phi^2(0) = 0$, then any $f \in V_0$ can be expressed in the form

$$f(x) = \sum_{\ell \in \mathbb{Z}} f(\ell)\phi^1(x - \ell) + f'(\ell)\phi^2(x - \ell), \tag{2.25}$$

for each $x \in \mathbb{R}$. Note that we can construct ϕ^1 and ϕ^2 as follows: Let $\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2 \in \mathscr{C}^1$ be defined by

$$\lambda_i^1 = \begin{cases} \lambda(3), & i = -1,\\ \lambda(1), & i = 0,\\ 0, & \text{otherwise}, \end{cases} \tag{2.26}$$

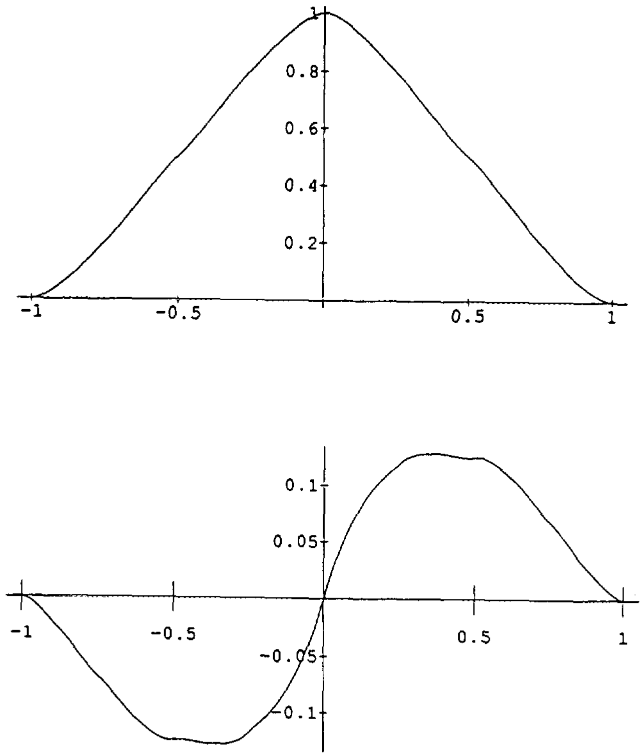

FIG. 1. Scaling functions from Example 1 with $s = \frac{3}{10}$.

and

$$\lambda_i^2 = \begin{cases} \lambda(4), & i = -1, \\ \lambda(2), & i = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.27}$$

Figure 1 shows the graphs of ϕ^1 and ϕ^2 for $s = \frac{3}{10}$.

We will continue this example in the next section. There it will be shown that ϕ^1 and ϕ^2 generate a multiresolution analysis on $L^2(\mathbb{R})$.

3. SCALING FUNCTION AND WAVELET EXPANSION

3.1. *MRAs of* $L^2(\mathbb{R})$

Let $\{\phi^\alpha\}_{\alpha=1}^A$ be a finite set of bounded and compactly supported functions in $L^2(\mathbb{R})$. We are interested in obtaining conditions which imply that

$\{\phi^\alpha\}_{\alpha=1}^A$ generates an MRA on $L^2(\mathbb{R})$. Let

$$V_0 = \overline{\operatorname{span}\{\phi^\alpha(\cdot - \ell): \alpha \in \{1, \dots, A\},\ \ell \in \mathbb{Z}\}}^{L^2(\mathbb{R})} \tag{3.1}$$

and

$$V_k = D_N^k V_0,$$

i.e., $f \in V_k \Leftrightarrow f(N^k \cdot) \in V_0$.

We first give sufficient conditions guaranteeing that the density (1.8) and separation (1.7) properties of the $\{V_k\}$ hold.

PROPOSITION 3.1. *Suppose that $\{\phi^\alpha\}_{\alpha=1}^A$ is a set of bounded and compactly supported functions in $L^2(\mathbb{R})$.*

1. *Separation.* $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$.
2. *Density. Suppose there exists an $a = (a_\alpha) \in \mathbb{R}^A$ such that*

$$\sum_{\alpha,\ell} a_\alpha \phi^\alpha(x - \ell) = 1, \tag{3.2}$$

for a.e. x in $\mathbb{R}$.

Then $\bigcup_{k\in\mathbb{Z}} V_k$ is dense in $L^2(\mathbb{R})$.

Proof. 1. Let $I_n = [n, n+1]$, $n \in \mathbb{Z}$, and let $U = \{f\chi_{I_0}: f \in V_0\}$. Since ϕ^α's are bounded and compactly supported U is a finite-dimensional linear space over $\mathbb{R}$, and therefore $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent on U. Hence there exists a positive constant c such that

$$\|f\|_\infty \le c\|f\|_2,$$

for all $f \in U$. By the translation invariance of V_0 we have

$$\|f\chi_{I_n}\|_\infty \le c\|f\chi_{I_n}\|_2,$$

for any $f \in V_0$. Thus

$$\|f\|_\infty = \sup_n \|f\chi_{I_n}\| \le c \sum_{n\in\mathbb{Z}} \|f\chi_{I_n}\|_2 = c\|f\|_2.$$

Following [23], note that

$$\|f\|_\infty \le cN^{-k/2}\|f_2\|,$$

for all $f \in V_k$. Hence if $f \in \bigcap V_k$, then $\|f\|_\infty = 0$.

2. Because of the translation and dilation invariance of $\bigcup_{k\in\mathbb{Z}} V_k$ it suffices to show that $\chi_{[0,1]} \in \overline{\bigcup_{k\in\mathbb{Z}} V_k}^{L^2(\mathbb{R})}$. Without loss of generality,

suppose that $\operatorname{supp}(\phi^\alpha) \subseteq [0, M]$. Let $j \in \mathbb{N}$ be such that $N^j > M$, and let $S_{N^j}(x) = \sum_{\ell=0}^{N^j} \sum_\alpha a_\alpha \phi^\alpha(x - \ell)$. Then $\operatorname{supp}(S_{N^j}) \subseteq [0, N^j + M]$, $S_{N^j}(x) = 1$ for $x \in [M, N^j]$, and $\|S_{N^j}\|_\infty \leq MA \max_\alpha \|a_\alpha \phi^\alpha\|_\infty$. It then follows that $\|S_{N^j}(N^j \cdot) - \chi_{[0,1]}\|_2 \to 0$. Since $S_{N^j}(N^j \cdot) \in \bigcup_{k \in \mathbb{Z}} V_k$, the result follows. ∎

Next we state necessary and sufficient conditions for the set of translates of the ϕ^α's to be a Riesz basis of V_0.

In what follows it is more convenient to use vector notation. Therefore, let

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi^2 \\ \vdots \\ \phi^A \end{pmatrix},$$

and

$$\mathbf{E}_\phi(\omega) := \sum_{k \in \mathbb{Z}} \hat{\boldsymbol{\phi}}(\omega + 2\pi k)\hat{\boldsymbol{\phi}}^*(\omega + 2\pi k), \tag{3.3}$$

where $\hat{\boldsymbol{\phi}}(\omega) := \int_{\mathbb{R}} e^{-i\omega x} \boldsymbol{\phi}(x)\, dx$ denotes the Fourier transform of $\boldsymbol{\phi}$ and * the Hermitian conjugate. Note that $\mathbf{E}_\phi$ is an $A \times A$ matrix. It is also easy to establish using the Poisson summation formula that

$$\left.\begin{aligned} \mathbf{E}_\phi(\omega) &= \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} \boldsymbol{\phi}(y-k)\boldsymbol{\phi}^*(y)\, dy \right) e^{i\omega k} \\ &= \sum_{k=-M+1}^{M-1} \left(\int_{\mathbb{R}} \boldsymbol{\phi}(y-k)\boldsymbol{\phi}^*(y)\, dy \right) e^{i\omega k} \end{aligned}\right\}, \tag{3.4}$$

where the last equality follows from the fact that $\boldsymbol{\phi}$ has compact support.

THEOREM 3.2. *The collection $\mathscr{B}_\phi = \{\phi^\alpha(\cdot - \ell): \alpha \in \{1, \dots, A\}, \ell \in \mathbb{Z}\}$ forms a Riesz basis for V_0 iff $\mathbf{E}_\phi(\omega)$ is nonsingular for $0 \leq \omega \leq 2\pi$.*

Proof. In vector notation (1.10) becomes

$$R_1 \|\{\mathbf{C}_\ell\}\|_2^2 \leq \int_{\mathbb{R}} \left| \sum_\ell \mathbf{C}_\ell^* \boldsymbol{\phi}(x - \ell) \right|^2 dx \leq R_2 \|\{\mathbf{C}_\ell\}\|_2^2, \tag{3.5}$$

for all $\{\mathbf{C}_\ell\} \in \ell^2(\mathbb{R}^A)$. By Parseval's Identity and the shift property of the

Fourier transform we have

$$\int_{\mathbb{R}}\left|\sum_{\ell}\mathbf{C}_\ell^*\boldsymbol{\phi}(x-\ell)\right|^2 dx = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{\ell}e^{i\omega\ell}\mathbf{C}_\ell^*\hat{\boldsymbol{\phi}}(\omega)\right|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\hat{\mathbf{C}}^*(\omega)\cdot\hat{\boldsymbol{\phi}}(\omega)\right|^2 d\omega, \qquad (3.6)$$

where $\hat{\mathbf{C}}(\omega) = \sum_{\ell\in\mathbb{Z}}e^{i\omega\ell}\mathbf{C}_\ell^*$. Since $\hat{\mathbf{C}}(\omega)$ is 2π-periodic, (3.6) becomes

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_0^{2\pi}\left|\hat{\mathbf{C}}^*(\omega)\hat{\boldsymbol{\phi}}(\omega+2\pi k)\right|^2 d\omega$$
$$= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_0^{2\pi}\left(\hat{\mathbf{C}}^*(\omega)\hat{\boldsymbol{\phi}}(\omega+2\pi k)\left(\hat{\boldsymbol{\phi}}^*(\omega+2\pi k)\hat{\mathbf{C}}(\omega)\right)\right) d\omega$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\hat{\mathbf{C}}^*(\omega)\mathbf{E}_\phi(\omega)\hat{\mathbf{C}}(\omega)\, d\omega$$

Again by Parseval's identity we have

$$\|\{\mathbf{C}_\ell\}\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}\hat{\mathbf{C}}^*(\omega)\hat{\mathbf{C}}(\omega)\, d\omega =: \frac{1}{2\pi}|||\hat{\mathbf{C}}|||_2^2.$$

Let $L^2([0,2\pi];\mathbb{C}^A) := \{\mathbf{f}\colon [0,2\pi]\to\mathbb{C}^A\colon\ |||\mathbf{f}|||_2 < \infty\}$. Therefore (3.5) is equivalent to

$$R_1|||\hat{\mathbf{C}}|||^2 \le \int_0^{2\pi}\hat{\mathbf{C}}^*(\omega)E_\phi(\omega)\hat{\mathbf{C}}(\omega)\, d\omega \le R_2|||\hat{\mathbf{C}}|||^2, \qquad (3.7)$$

for all $\hat{\mathbf{C}} \in L^2([0,2\pi];\mathbb{C}^A)$.

From (3.3) it is clear that $\mathbf{E}_\phi(\omega)$ is self-adjoint and positive, and thus has real nonnegative eigenvalues $\lambda_\alpha(\omega)$, $\alpha = 1,\ldots,A$. The second equality in (3.4) shows that $\mathbf{E}_\phi(\omega)$ is continuous in ω and therefore the eigenvalues are also continuous. Let $a(\omega) = \min_\alpha \lambda_\alpha(\omega)$ and $b(\omega) = \max_\alpha \lambda_\alpha(\omega)$. Then

$$a(\omega)\hat{\mathbf{C}}^*(\omega)\hat{\mathbf{C}}(\omega) \le \hat{\mathbf{C}}^*(\omega)\mathbf{E}_\phi(\omega)\hat{\mathbf{C}}(\omega) \le b(\omega)\hat{\mathbf{C}}^*(\omega)\hat{\mathbf{C}}(\omega)$$

and so (3.7) holds with

$$R_1 = \min\{a(\omega)\colon \omega\in[0,2\pi]\} \quad\text{and}\quad R_2 = \max\{b(\omega)\colon \omega\in[0,2\pi]\}.$$

In fact, since (3.7) holds for all $\hat{\mathbf{C}} \in L^2([0,2\pi];\mathbb{C}^A)$, these are the best possible bounds. Since $\max_\omega b(\omega)$ is always finite, we conclude that $\mathscr{B}_\phi$ is a Riesz basis if and only if $\min_\omega a(\omega) > 0$ which is true if and only if $\mathbf{E}_\phi(\omega)$ is nonsingular for $0 \leq \omega \leq 2\pi$. ∎

A necessary and sufficient condition for $V_1 \subseteq V_0$ is that each ϕ^α be a linear combination of the translates of $\phi^{\alpha'}(N\,\cdot\,)$, $\alpha' = 1,\ldots,A$. More precisely, $V_1 \subseteq V_0$ if and only if $\boldsymbol{\phi}$ satisfies a two-scale dilation equation of the form

$$\boldsymbol{\phi}(x) = \sum_{\ell \in \mathbb{Z}} \mathbf{p}_\ell \boldsymbol{\phi}(Nx - \ell), \tag{3.8}$$

for some sequence of $A \times A$ matrices $\{\mathbf{p}_\ell\}$. We remark that the compact support of ϕ implies that $\mathbf{p}_\ell = 0$ for all but a finite number of ℓ's.

EXAMPLE 1—*Continued*. Using (2.17) we can calculate ϕ^i and $(\phi^i)'$, $i = 1,2$, at $x = \pm\frac{1}{2}$. Equation (2.25) then gives us

$$\left.\begin{aligned} \mathbf{p}_{-1} &= \begin{pmatrix} \frac{1}{2} & 1 - 2s \\ -\frac{1}{8} & s - \frac{1}{4} \end{pmatrix} \\ \mathbf{p}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ \mathbf{p}_1 &= \begin{pmatrix} \frac{1}{2} & 2s - 1 \\ \frac{1}{8} & \frac{1}{4} - s \end{pmatrix} \end{aligned}\right\}. \tag{3.9}$$

A set consisting of $B = A(N-1)$ functions $\psi^1,\ldots,\psi^B$ with the property that $\mathscr{B}_\psi = \{\psi^\beta(\cdot - \ell)\colon \beta = 1,\ldots,B;\ \ell \in \mathbb{Z}\}$ constitutes a Riesz basis of the orthogonal complement W_0 of V_0 in V_{-1} is called a set of *wavelets* associated with the scaling functions $\phi^1,\ldots,\phi^A$. The orthogonal complement of V_k in V_{k-1} is denoted by W_k. It is easy to see that the wavelet spaces $\{W_k\}_{k\in\mathbb{Z}}$ give an orthogonal direct sum decomposition of $L^2(\mathbb{R})$.

Since $W_0 \subset V_{-1}$ there exists a sequence of $B \times A$ matrices $\{\mathbf{q}_\ell\}_{\ell\in\mathbb{Z}}$ such that

$$\boldsymbol{\psi}(x/N) = \sum_\ell \mathbf{q}_\ell \boldsymbol{\phi}(x - \ell) \tag{3.10}$$

and $\sum_{\beta',\ell} |q_\ell^{\beta\beta'}|^2 < \infty$ for $\beta = 1,\ldots,B$, where

$$\boldsymbol{\psi} = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^B \end{pmatrix}.$$

Applying Theorem 3.2 with $\boldsymbol{\phi}$ replaced by $\boldsymbol{\psi}$ shows that $\mathscr{B}_\psi$ is a Riesz basis of W_0 iff

$$\text{(i) } W_0 \subset \overline{\operatorname{span} \mathscr{B}_\psi}^{L^2(\mathbb{R})}; \tag{3.11}$$

$$\text{(ii) } \mathbf{E}_\psi(\omega) := \sum_{k\in\mathbb{Z}} \hat{\boldsymbol{\psi}}(\omega + 2\pi k)\hat{\boldsymbol{\psi}}^*(\omega + 2\pi k) \text{ is nonsingular for } \omega \in [0, 2\pi]. \tag{3.12}$$

We next derive a relation between $\mathbf{E}_\phi$ and $\mathbf{E}_\psi$ using (3.10).

THEOREM 3.3. *Let* $z = e^{-i\omega/N}$ *and let* $Q(z) = (1/N)\Sigma_\ell \mathbf{q}_\ell z^\ell$.

$$(a)\ \mathbf{E}_\psi(\omega) = \sum_{j=0}^{N-1} Q(e^{-i(2\pi i/N)}z)\mathbf{E}_\phi\left(\frac{\omega + 2\pi j}{N}\right)Q(e^{-i(2\pi i/N)}z)^*. \tag{3.13}$$

(*b*) *Suppose* $\mathbf{E}_\phi(\omega)$ *is nonsingular for all* ω. *Then* $\mathbf{E}_\psi(\omega)$ *is nonsingular for all* ω *if*

$$\bigcup_{j=0}^{N-1} \operatorname{null}\left(Q(e^{-i(2\pi i/N)}z)^*\right) = \{0\},$$

where null denotes null space.

Remark. In the case $N = 2$, we have $A = B$ and condition (b) will hold if $\det(Q(z))$ has no N-symmetric zeros on the unit circle. (A complex number $z \neq 0$ is called an *N-symmetric zero* of a function $f\colon \mathbb{C} \to \mathbb{C}$ if and only if $f(e^{(-i2\pi i/N)}z) = 0$, $j = 0, \ldots, N-1$.)

Proof. (a) Using (3.10) and (3.12), we have

$$\begin{aligned}
\mathbf{E}_\psi(\omega) &= \sum_{k\in\mathbb{Z}}\left(\frac{1}{N}\sum_\ell \mathbf{q}_\ell e^{-i((\omega+2\pi k)/N)\ell}\hat{\boldsymbol{\phi}}\left(\frac{\omega + 2\pi k}{N}\right)\right) \\
&\quad \times \sum_\ell \left(\frac{1}{N}\mathbf{q}_\ell e^{-i((\omega+2\pi k)/N)\ell}\hat{\boldsymbol{\phi}}\left(\frac{\omega + 2\pi k}{N}\right)\right)^* \\
&= \sum_{j=0}^{N-1} Q(e^{-i(\omega+2\pi j)/N})\left(\sum_{k'\in\mathbb{Z}} \hat{\boldsymbol{\phi}}\left(\frac{\omega + 2\pi j}{N} + 2\pi k'\right)\right. \\
&\quad \left. \cdot \hat{\boldsymbol{\phi}}^*\left(\frac{\omega + 2\pi j}{N} + 2\pi k'\right)\right)Q(e^{-i(\omega+2\pi j)/N})^*,
\end{aligned}$$

where $k = Nk' + j$. The result now follows.

(b) Since $Q(z)\mathbf{E}_\phi(\omega)Q(z)^*$ is a positive matrix for all ω, it is clear from (3.13) that $\mathbf{E}_\psi(\omega)$ is nonsingular if

$$\bigcup_{j=0}^{N-1} \text{null}\left(Q(e^{-i(2\pi i/N)}z)\mathbf{E}_\phi((\omega + 2\pi j)/N)Q(e^{-i(2\pi i/N)}z)^*\right)$$

is trivial. By the hypothesis,

$$\begin{aligned}\text{null}\left(Q(e^{-i(2\pi i/N)}z)\mathbf{E}_\phi(\omega + 2\pi j/N)Q(e^{-i(2\pi i/N)}z)^*\right) \\ = \text{null}\left(Q(e^{-i(2\pi i/N)}z)^*\right)\end{aligned}$$

and the conclusion follows. ∎

We next show that the C^1 scaling functions constructed in Section 2.3 generate an MRA on $L^2(\mathbb{R})$.

3.2. *Example* 1—*Continued*

Let ϕ^1 and ϕ^2 be again defined by (2.26) and (2.27). Let

$$V_0 = \overline{\text{span}\{\phi^i(\cdot - \ell): i = 1,2;\ \ell \in \mathbb{Z}\}}^{L^2(\mathbb{R})}.$$

Note that $V_0 = (\tau \circ \boldsymbol{\theta})(\Pi_2^1) \cap L^2(\mathbb{R})$ and hence $V_1 = D_2 V_0 \subseteq V_0$. Therefore, the nestedness condition (1.7) holds.

Since $\lambda(1) + \lambda(3) = (1 - s, 0, 0, 1 - s, 0, 0)$, it follows from (2.2) that $\theta(\lambda(1) + \lambda(3)) \equiv 1$. Thus ϕ^1 forms a partition of unity

$$\sum_{\ell \in \mathbb{Z}} \phi^1(x - \ell) \equiv 1.$$

As ϕ^1 and ϕ^2 are bounded, compactly supported, and satisfy condition (3.2) with $a = (1,0)$, it follows from Proposition 3.1 that the separation and density properties of $\{V_k\}$ hold.

Finally, we show that ϕ^1, ϕ^2, and their integer-translates form a Riesz basis of V_0. By Theorem 3.2 it suffices to show that $\mathbf{E}_\phi(\omega)$ is nonsingular, for all $\omega \in [0, 2\pi]$. For our example, the number M in (3.4) is equal to 2 since the support for both ϕ^1 and ϕ^2 is the interval $[-1, 1]$.

Let $e_k := \int_{\mathbb{R}} \phi(y-k)\phi^*(y)\,dy$, $k = 0, \pm 1$. The e_k's are calculated using (A.1) in the Appendix and Mathematica. They are given by

$$e_0 = \begin{pmatrix} \dfrac{92 - 69s - 78s^2 + 56s^3}{60(2-s)(1-s^2)} & 0 \\ 0 & \dfrac{5-4s}{240(1-s^2)} \end{pmatrix},$$

$$e_1 = \begin{pmatrix} \dfrac{28 + 9s - 42s^2 + 4s^3}{120(2-s)(1-s^2)} & \dfrac{13 + 9s - 27s^2 + 4s^3}{240(2-s)(1-s^2)} \\ \dfrac{-13 - 9s + 27s^2 - 4s^3}{240(s-2)(1-s^2)} & \dfrac{-3 - 4s + 8s^2}{480(1-s^2)} \end{pmatrix},$$

and $e_{-1} = e_1^T$.

Hence $\mathbf{E}_\phi(\omega) = e_{-1}e^{-i\omega} + e_0 + e_1 e^{i\omega}$, $\omega \in [0, 2\pi]$, and therefore,

$$\det\big(\mathbf{E}_\phi(\omega)\big) = \frac{\gamma_0 + \gamma_1 z + \gamma_2 z^2 + \gamma_1 z^3 + \gamma_0 z^4}{57{,}600(s-2)^2(s^2-1)z^2}, \tag{3.14}$$

where $z = e^{i\omega}$ and

$$\gamma_0 = -1 - 40s - 147s^2 + 200s^3 - 16s^4,$$

$$\gamma_1 = 544 + 640s - 3912s^2 + 3520s^3 - 896s^4,$$

$$\gamma_2 = -3006 + 8400s - 7722s^2 + 2160s^3 - 96s^4.$$

Note that the denominator in (3.14) does not vanish for $|s| < 1$. The roots of the quartic equation in the numerator of (3.14) are of the form

$$\rho \pm \sqrt{\rho^2 - 1}, \tag{3.15}$$

where

$$\rho = -(\gamma_1/4\gamma_0) \pm \left(\sqrt{2 + (\gamma_1/2\gamma_0)^2 - (\gamma_2/\gamma_0)}\right)/2,$$

assuming $\gamma_0 \neq 0$.

It follows from (3.15) that the zeros of $\det(\mathbf{E}_\phi(\omega))$ are on the unit circle if and only if ρ is real and $\rho^2 \leq 1$. However, this is equivalent to

$$\mathscr{R}(s) := (2\gamma_0 + \gamma_2)^2 - 4\gamma_1^2 \leq 0.$$

The polynomial $\mathscr{R}(s)$ factors into

$$30720(2-s)^3(2s-1)^2(32-39s-18s^2+26s^3).$$

Using methods from elementary calculus, one can easily show that the above cubic polynomial is strictly positive for $|s|<1$. Hence $\mathscr{R}(s)>0$, except for $s=1/2$.

Therefore, $\boldsymbol{\phi}$ generates an MRA on $L^2(\mathbb{R})$, for any $s\in(-1,1)\setminus\{\frac{1}{2}\}$. If $|s|<\frac{1}{2}$, ϕ^1 and ϕ^2 are also in $C^1(\mathbb{R})$. Next we construct the wavelets. As we will see, it is possible to choose wavelets whose support is $[-1,2]$. This corresponds to choosing $\mathbf{q}_\ell=0$, for $\ell\notin\{-1,0,1,2,3\}$ in (3.10). Let $\psi\in V_{-1}$ be given by

$$\psi(x)=\sum_{\ell=-1}^{3}\sum_{j=0}^{1}c_\ell^j\phi^j(2x-\ell), \tag{3.16}$$

for some $c_\ell^j\in\mathbb{R}$. A necessary and sufficient condition for $\psi\in W_0\subseteq V_{-1}$ is

$$\langle\phi^i(\cdot-k),\psi\rangle=0 \tag{3.17}$$

for $i=0,1$ and $k=-1,0,1,2$ (that we only need to consider $k=-1,0,1,2$ in (3.17) follows from the supports of ϕ^i and ψ). Using (3.8), (3.9), and (3.16), condition (3.17) may be written as

$$\sum_{\ell=-1}^{3}\sum_{\ell'=-1}^{1}\sum_{j,j'=0}^{1}p_{\ell'}^{i,j'}\langle\phi^{j'}(2x-2k-\ell'),\phi^j(2x-\ell)\rangle c_\ell^j=0 \tag{3.18}$$

for $i=0,1$ and $k=-1,0,1,2,3$, where $\mathbf{p}_\ell=(p_\ell^{i,j})$. Note that (3.18) is a linear system of eight equations in the ten unknowns c_ℓ^j. Using Mathematica, we obtain two solutions $\mathbf{c}^1=(q_\ell^{0,j})$ and $\mathbf{c}^2=(q_\ell^{1,j})$ that form a basis of the nullspace of the linear system (3.18). Let $\mathbf{q}_\ell=(q_\ell^{i,j})_{i,j=0,1}$. Then, for $s=\frac{3}{10}$, the $\mathbf{q}_\ell$'s are approximately

$$\mathbf{q}_{-1}=\begin{pmatrix}-0.297 & -1.328\\ 0 & -0.0064\end{pmatrix}$$

$$\mathbf{q}_0=\begin{pmatrix}-0.436 & -7.898\\ 0.1934 & 0.7801\end{pmatrix}$$

$$\mathbf{q}_1=\begin{pmatrix}0.990 & 6.969\\ 0.513 & 6.969\end{pmatrix}$$

$$\mathbf{q}_2=\begin{pmatrix}-0.026 & 4.344\\ -0.425 & 6.937\end{pmatrix}$$

$$\mathbf{q}_3=\begin{pmatrix}-0.230 & 1.217\\ -0.282 & 1.464\end{pmatrix}.$$

Then defining $\boldsymbol{\psi} = (\psi^1, \psi^2)^T$ by (3.10) we have $\psi^1, \psi^2 \in W_0$. The results obtained in the next section will imply that $\overline{\operatorname{span} \mathscr{B}_\psi}^{L^2(\mathbb{R})} = W_0$.

Here we show that $\mathscr{B}_\psi$ is a Riesz basis for the L^2-closure of its span. The Q-symbol for $\boldsymbol{\psi}$ is given by $Q(z) = \frac{1}{2}(\mathbf{q}_{-1}z^{-1} + \mathbf{q}_0 + \mathbf{q}_1 z + \mathbf{q}_2 z^2 + \mathbf{q}_3 z^3)$. Using the exact values for the $\mathbf{q}_\ell$'s one can show by elementary, although tedious, calculations that $\det(Q(z))$ has no 2-symmetric zeros on the unit circle. Figure 2 shows the wavelets ψ^1 and ψ^2 for $s = \frac{3}{10}$.

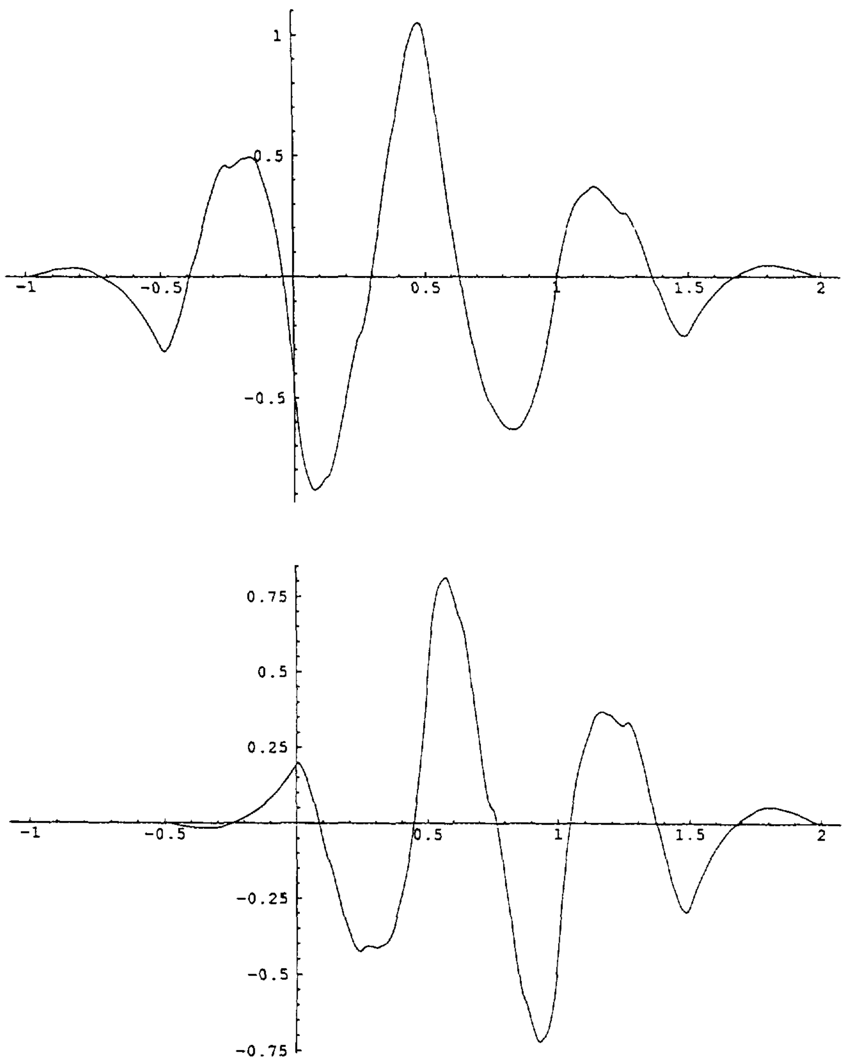

FIG. 2. Wavelets from Example 1 with $s = \frac{3}{10}$.

4. Decomposition and Reconstruction Algorithms

Suppose we are given a function $f_0 \in V_0$. Since $V_0 = V_1 \oplus W_1$, we can uniquely decompose f_0 as

$$f_0 = f_1 + g_1, \tag{4.1}$$

where $f_1 \in V_1$ and $g_1 \in W_1$. Conversely, given an $f_1 \in V_1$ and $g_1 \in W_1$, one can reconstruct f_0. Since $f_0 \in V_0$ there exists a sequence of vectors $\mathbf{c}(0) = \{c_\ell(0)\}_{\ell \in \mathbb{Z}} \in \ell^2(\mathbb{Z})^A$ such that

$$f_0(x) = \sum_{\ell \in \mathbb{Z}} c_\ell^*(0)\boldsymbol{\phi}(x - \ell). \tag{4.2}$$

Similarly, there exist vector sequences $\mathbf{c}(1) = \{c_\ell(1)\} \in \ell^2(\mathbb{Z})^A$ and $\mathbf{d}(1) = \{d_\ell(1)\} \in \ell^2(\mathbb{Z})^B$ so that

$$f_1(x) = \sum_{\ell \in \mathbb{Z}} c_\ell^*(1)\boldsymbol{\phi}\left(\frac{x}{N} - \ell\right), \tag{4.3}$$

and

$$g_1(x) = \sum_{\ell \in \mathbb{Z}} d_\ell^*(1)\boldsymbol{\psi}\left(\frac{x}{N} - \ell\right). \tag{4.4}$$

Using (3.8) and (3.10) we obtain the following reconstruction algorithm:

$$c_\ell(0) = \sum_{\ell' \in \mathbb{Z}} c_{\ell'}^*(1)\mathbf{p}_{\ell - N\ell'} + d_{\ell'}^*(1)\mathbf{q}_{\ell - N\ell'}, \tag{4.5}$$

for $\ell \in \mathbb{Z}$.

Note that this algorithm is finite if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are compactly supported.

If $\mathscr{B}_\phi$ and $\mathscr{B}_\psi$ are orthogonal systems then the decomposition algorithm is easy to obtain using (3.8) and (3.9). To deal with the case that $\mathscr{B}_\phi$ and $\mathscr{B}_\psi$ are not orthogonal systems one has to introduce the *dual bases* $\{\tilde{\phi}_\ell^\alpha$: $\alpha = 1, \ldots, A$; $\ell \in \mathbb{Z}\}$ and $\{\tilde{\psi}_\ell^\beta$: $\beta = 1, \ldots, B$; $\ell \in \mathbb{Z}\}$.

Theorem 4.1. *Suppose that $\mathscr{B}_\psi$ is a Riesz basis for V_0. Let $\tilde{\boldsymbol{\phi}} \in L^2(\mathbb{R})^A$ be defined by*

$$\hat{\tilde{\boldsymbol{\phi}}}(\omega) := \mathbf{E}_\phi^{-1}(\omega)\hat{\boldsymbol{\phi}}(\omega), \tag{4.6}$$

for $\omega \in \mathbb{R}$. Then $\tilde{\phi}^\alpha \in V_0$, $\alpha = 1, \ldots, A$, and

$$\sum_{k \in \mathbb{Z}} \hat{\boldsymbol{\phi}}(\omega + 2\pi k)\hat{\tilde{\boldsymbol{\phi}}}^*(\omega + 2\pi k) = I. \tag{4.7}$$

Furthermore,

$$\langle\langle \boldsymbol{\phi}_\ell, \tilde{\boldsymbol{\phi}}_k \rangle\rangle := \int_{\mathbb{R}} \boldsymbol{\phi}_\ell(x) \tilde{\boldsymbol{\phi}}_k^*(x)\, dx = \delta_{\ell k} I, \tag{4.8}$$

for all ℓ, $k \in \mathbb{Z}$, *where* $\boldsymbol{\phi}_\ell = \boldsymbol{\phi}(\cdot - \ell)$.

Proof. Since $\mathscr{B}_\phi$ is a Riesz basis the proof of Theorem 3.2 shows that the eigenvalues $\lambda(\omega)$ of $\mathbf{E}_\phi(\omega)$ satisfy

$$0 < R_1 \le \lambda(\omega) \le R_2 < \infty,$$

for constants R_1 and R_2 independent of $\omega \in [0, 2\pi]$. Therefore, the eigenvalues of $\mathbf{E}_\phi^{-1}(\omega)$ are bounded by $1/R_2$ and $1/R_1$. It follows from the self-adjointness of $\mathbf{E}_\phi^{-1}(\omega)$ that its coefficients are bounded and in $L^2([0, 2\pi])$. Thus, $\mathbf{E}_\phi^{-1}(\omega)$ has a Fourier series expansion of the form

$$\mathbf{E}_\phi^{-1}(\omega) = \sum_{n \in \mathbb{Z}} \mathbf{e}_n e^{-in\omega}, \tag{4.9}$$

where $\{\mathbf{e}_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})^A$. From (4.6) and (4.9) we get

$$\tilde{\boldsymbol{\phi}}(x) = \sum_{\ell \in \mathbb{Z}} \mathbf{e}_\ell \boldsymbol{\phi}(x - \ell), \tag{4.10}$$

which shows that $\tilde{\boldsymbol{\phi}} \in V_0$.

The identity (4.7) follows directly from (4.6) and the fact that $\mathbf{E}_\phi(\omega)$ is 2π-periodic. By Parseval's Identity and (4.7) we have that

$$\begin{aligned}
\langle\langle \boldsymbol{\phi}_\ell, \tilde{\boldsymbol{\phi}}_k \rangle\rangle &= (1/2\pi) \int_{\mathbb{R}} e^{i\omega(\ell - k)} \hat{\boldsymbol{\phi}}(\omega) \hat{\tilde{\boldsymbol{\phi}}}^*(\omega)\, d\omega \\
&= (1/2\pi) \sum_{j \in \mathbb{Z}} \int_0^{2\pi} e^{i\omega(\ell - k)} \hat{\boldsymbol{\phi}}(\omega + 2\pi j) \hat{\tilde{\boldsymbol{\phi}}}^*(\omega + 2\pi j)\, d\omega \\
&= (1/2\pi) \int_0^{2\pi} e^{i\omega(\ell - k)} \sum_{j \in \mathbb{Z}} \hat{\boldsymbol{\phi}}(\omega + 2\pi j) \hat{\tilde{\boldsymbol{\phi}}}^*(\omega + 2\pi j)\, d\omega \\
&= \delta_{\ell k} I,
\end{aligned}$$

for all k, $\ell \in \mathbb{Z}$. ▮

Let $\{\psi^1, \ldots, \psi^B\}$ be a set of associated wavelets, i.e., $\mathscr{B}_\psi$ is a Riesz basis for W_0, the orthogonal complement of V_0 in V_{-1}. Then Theorem 4.1 implies the existence of a dual vector $\tilde{\boldsymbol{\psi}} = \{\tilde{\boldsymbol{\psi}}^\beta\}_{\beta=1}^B$ whose components are in W_0 and which satisfies

$$\langle\langle \boldsymbol{\psi}_\ell, \tilde{\boldsymbol{\psi}}_k \rangle\rangle = \delta_{\ell k} I, \tag{4.11}$$

for all $k, \ell \in \mathbb{Z}$.

Since $W_0 \perp V_0$, we also have

$$\langle\langle \boldsymbol{\phi}_\ell, \tilde{\boldsymbol{\psi}}_k \rangle\rangle = 0 \tag{4.12}$$

and

$$\langle\langle \tilde{\boldsymbol{\phi}}_\ell, \boldsymbol{\psi}_k \rangle\rangle = 0, \tag{4.13}$$

for all $k, \ell \in \mathbb{Z}$.

Let $Q(z)$ be as in Theorem 3.3 and let

$$P(z) = \frac{1}{N} \sum_{\ell \in \mathbb{Z}} \mathbf{p}_\ell z^\ell. \tag{4.14}$$

Define

$$G(z) = \mathbf{E}_{\tilde{\phi}}^{-1}(z^N) P(z) \mathbf{E}_\phi(z) \tag{4.15}$$

for $z = e^{i\omega/N}$, where we abuse notation and write $\mathbf{E}_\phi(z)$ for $\mathbf{E}_\phi(\omega/N)$. Since $\boldsymbol{\phi}$ is compactly supported both $P(z)$ and $\mathbf{E}_\phi(z)$ are Laurent polynomials in z. As $\mathbf{E}_\phi$ is nonsingular on the unit circle, it follows that $G(z)$ is analytic on $|z| = 1$, and therefore has a Laurent series of the form

$$G(z) = \frac{1}{N} \sum_{\ell \in \mathbb{Z}} \mathbf{g}_\ell z^\ell$$

with exponentially decaying matrix coefficients $\mathbf{g}_\ell$.

From (4.6) it easily follows that

$$\hat{\tilde{\boldsymbol{\phi}}}(\omega) = G(e^{i\omega}) \hat{\tilde{\boldsymbol{\phi}}}\left(\frac{\omega}{N}\right)$$

or, equivalently,

$$\tilde{\boldsymbol{\phi}}(x) = \sum_{\ell \in \mathbb{Z}} \mathbf{g}_\ell \tilde{\boldsymbol{\phi}}(Nx - \ell). \tag{4.16}$$

Similarly, if we define

$$H(z) = \mathbf{E}_{\tilde{\psi}}^{-1}(z^N) Q(z) \mathbf{E}_\phi(z), \tag{4.17}$$

then there are exponential decaying coefficients $\mathbf{h}_\ell$ such that

$$\tilde{\boldsymbol{\psi}}(x) = \sum_{\ell \in \mathbb{Z}} \mathbf{h}_\ell \tilde{\boldsymbol{\phi}}(Nx - \ell). \tag{4.18}$$

THEOREM 4.2. *Let $\boldsymbol{\phi} = \{\phi^\alpha\}_{\alpha=1}^A$ generate an MRA on $L^2(\mathbb{R})$ with associated wavelets $\boldsymbol{\psi}$. Let $\{\mathbf{g}_\ell\}$ and $\{\mathbf{h}_\ell\}$ be as in* (4.16) *and* (4.18), *respectively. Then*

$$\boldsymbol{\phi}(Nx - \ell) = \sum_{k \in \mathbb{Z}} \mathbf{g}^*_{\ell - Nk} \boldsymbol{\phi}(x - k) + \mathbf{h}^*_{\ell - Nk} \boldsymbol{\psi}(x - k), \quad (4.19)$$

for all $\ell \in \mathbb{Z}$.

Proof. Since $\phi^\alpha(N \cdot - \ell) \in V_{-1} = V_0 \oplus W_0$, for $\alpha = 1, \ldots, A$ and $\ell = 0, \ldots, N-1$, we have $\boldsymbol{\phi}(Nx - \ell) = \sum_{k \in \mathbb{Z}} \mathbf{a}_{\ell - Nk} \boldsymbol{\phi}(x - k) + b_{\ell - Nk} \boldsymbol{\psi}(x - k)$, for some matrix coefficients $\{\mathbf{a}_\ell\}$ and $\{\mathbf{b}_\ell\}$. Applying $\langle\langle \tilde{\boldsymbol{\phi}} \cdot \rangle\rangle$ and $\langle\langle \tilde{\boldsymbol{\psi}}, \cdot \rangle\rangle$ to the above equation yields for result. ∎

Now we are ready to state the decomposition algorithm.

Let $f_0 \in V_0 = V_1 \oplus W_1$, and let f_1 and g_1 be the unique functions in V_1 and W_1, respectively, such that

$$f_0 = f_1 + g_1.$$

Let $\mathbf{c}(0)$, $\mathbf{c}(1)$, and $\mathbf{d}(1)$ be as in (4.2), (4.3), and (4.4). Then, using (4.19), we obtain

$$c_k(1) = \sum_{\ell \in \mathbb{Z}} \mathbf{g}_{\ell - Nk} c_\ell(0), \quad (4.20)$$

and

$$d_k(1) = \sum_{\ell \in \mathbb{Z}} \mathbf{h}_{\ell - Nk} c_\ell(0). \quad (4.21)$$

EXAMPLE 1—*Continued.* We calculate $G(z)$ using the setup in Example 1. For the sake of simplicity we choose $s = \frac{3}{10}$. Using formula (4.15), we find that the four entries in the matrix $G(z)$ are rational functions in z having the same denominator. The zeros of this denominator which is given by

$$52{,}399 - 117{,}925 z^2 + 2{,}808{,}594 z^4 - 117{,}925 z^6 + 52{,}399 z^8$$

are the same as the zeros of $\det(\mathbf{E}_\phi(z^2))$ (all the calculations are done in Mathematica). These zeros can be calculated exactly, however, due to the complexity of their expressions, we have opted to give only their approximate numerical values. These are

$$z_1 = 4.45689 = -z_2,$$
$$z_3 = 1.45577 = -z_4,$$
$$z_5 = 0.224372 = -z_6,$$
$$z_7 = 0.686922 = -z_8.$$

Since the numerators in $G(z)$ have the same degree as the denominators, we may express $G(z)$ as a partial fraction expansion of the form

$$G(z) = \mathbf{a}_0 + \sum_{j=1}^{8} \frac{\mathbf{a}_j}{z - z_j}, \tag{4.22}$$

where

$$\mathbf{a}_0 = \begin{pmatrix} 0.908426 & -0.215307 \\ 2.05031 & -0.458426 \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} 2.9597 & 0.258477 \\ -14.9584 & 1.30635 \end{pmatrix},$$

$$\mathbf{a}_2 = \begin{pmatrix} 2.1651 & 0.657654 \\ 10.9425 & 3.32381 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0.0460402 & -0.00690382 \\ 0.842373 & -0.126315 \end{pmatrix},$$

$$\mathbf{a}_4 = \begin{pmatrix} -0.558525 & -0.168794 \\ -10.219 & 3.08842 \end{pmatrix}, \quad \mathbf{a}_5 = \begin{pmatrix} 0.0401392 & -0.00118983 \\ -0.202865 & 0.00601343 \end{pmatrix},$$

$$\mathbf{a}_6 = \begin{pmatrix} -0.000136971 & 0.0721497 \\ 0.000692257 & -0.364648 \end{pmatrix}, \quad \mathbf{a}_7 = \begin{pmatrix} -0.0107273 & -0.00338801 \\ 0.196271 & 0.0619885 \end{pmatrix},$$

$$\mathbf{a}_8 = \begin{pmatrix} 0.0659662 & -4.62076 \\ -1.20695 & -0.126315 \end{pmatrix}.$$

To obtain the $\{\mathbf{g}_\ell\}_{\ell \in \mathbb{Z}}$, we expand (4.22) into a Laurent series converging in an annulus containing the unit circle. This gives

$$\mathbf{g}_0 = \mathbf{a}_0 - \sum_{j=1}^{4} \frac{\mathbf{a}_j}{z_j}$$

and

$$\mathbf{g}_\ell = \begin{cases} \displaystyle\sum_{j=5}^{8} \mathbf{a}_j z_j^{-(\ell+1)}, & \ell > 0, \\ \displaystyle -\sum_{j=1}^{4} \mathbf{a}_j z_j^{-(\ell+1)}, & \ell < 0. \end{cases}$$

We remark that $\mathbf{g}_\ell = \mathcal{O}(z_7^{-|\ell|})$.

In a similar fashion one can calculate the $\{\mathbf{h}_\ell\}_{\ell \in \mathbb{Z}}$.

5. Example 2: Orthonormal Scaling Functions

In this section we construct scaling functions ϕ^1 and ϕ^2 such that the set $\mathscr{B}_\phi$ of their integer-translates in an orthonormal system. In [11], we use these scaling functions to construct two compactly supported and continuous wavelets ψ^1 and ψ^2 so that the set $\mathscr{B}_\psi$ is also orthonormal. Such an orthonormal system is useful in applications where a function $f \in L^2(\mathbb{R})$ is first projected onto V_0. This example also illustrates how free parameters such as s can be used to construct scaling functions and wavelets satisfying certain conditions.

Let $N = 2$ and choose s in $(-1, 1)$. Let $f_{(a,b,c;s)}$ be the unique continuous fractal function on $[0, 1]$ generated by affine λ's interpolating the set $\{(0, a), (\frac{1}{2}, b), (1, c)\}$, where $a, b, c \in \mathbb{R}$. Let ϕ^1 and ϕ^2 be defined by

$$\phi^1(x) = \begin{cases} f_{(0,1,0;s)}(x), & x \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\phi^2(x) = \begin{cases} f_{(0,a,1;s)}(x), & x \in [0,1], \\ f_{(1,a,0;s)}(x), & x \in [1,2], \\ 0, & \text{otherwise}. \end{cases}$$

We determine a and s such that $\mathscr{B}_\phi$ is an orthogonal system. Due to the symmetry and supports of ϕ^1 and ϕ^2 it suffices to have

$$\langle f_{(0,1,0;s)}, f_{(0,a,1;s)} \rangle = 0 \tag{5.1}$$

and

$$\langle f_{(0,a,1;s)}, f_{(1,a,0;s)} \rangle = 0 \tag{5.2}$$

Using (A.2), we obtain from (5.1) that $a = (3s^2 + s - 1)/(2s + 4)$. Condition (5.2) then yields $s = -1/5$. The graphs of ϕ^1 and ϕ^2 are shown in Fig. 3. Using the values of ϕ^1 and ϕ^2 at $n/4$, $n = 0, \ldots, 8$, one can solve for the matrix coefficients in the dilation equation (3.8). These are

$$\mathbf{p}_0 = \begin{pmatrix} 3/5 & 4\sqrt{2/5} \\ -1/10\sqrt{2} & -3/10 \end{pmatrix}, \qquad \mathbf{p}_1 = \begin{pmatrix} 3/5 & 0 \\ 9/10\sqrt{2} & 1 \end{pmatrix},$$

$$\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 9/10\sqrt{2} & -3/10 \end{pmatrix}, \qquad \mathbf{p}_3 = \begin{pmatrix} 0 & 0 \\ -1/10\sqrt{2} & 0 \end{pmatrix}.$$

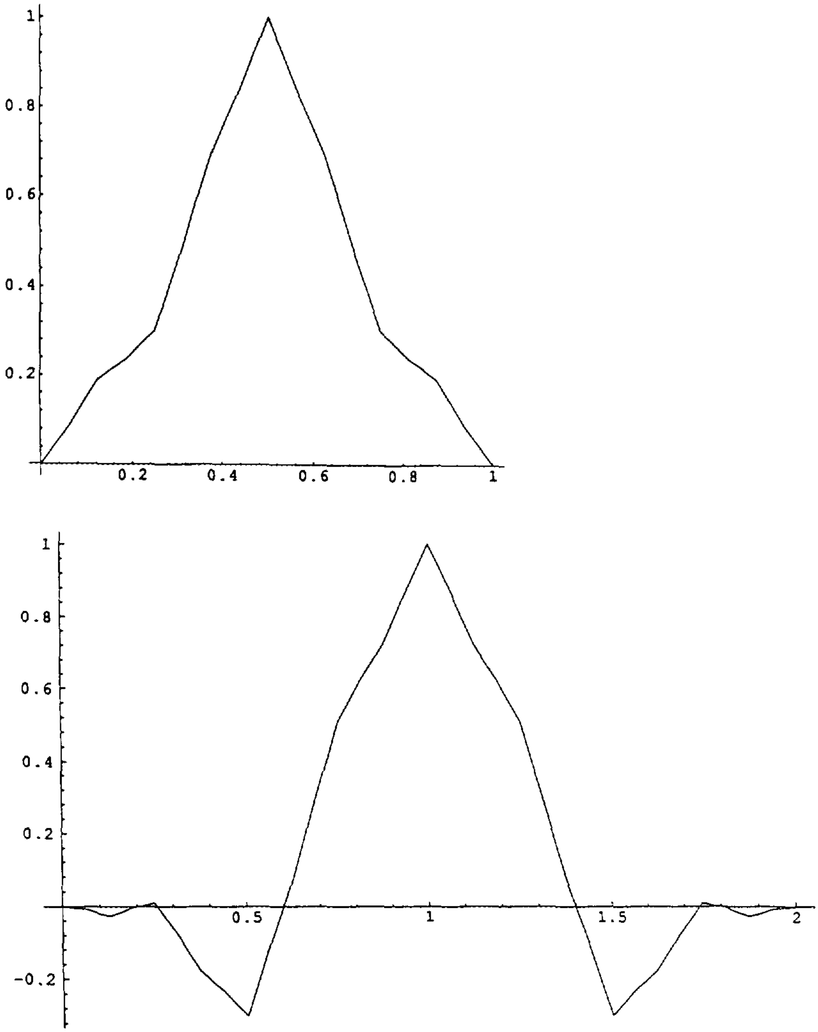

FIG. 3. The orthogonal scaling functions from Example 2.

APPENDIX

Let $\lambda, \mu \in \mathscr{B}$ and let f and g, respectively, be the associated fractal functions, i.e., $f = \Phi_\lambda(f)$ and $g = \Phi_\mu(g)$ (cf. (2.2)). Define the inner product of f and g over I by

$$\langle f, g \rangle := \int_I f(x) g(x)\, dx.$$

Then using the fixed point property of f and g we have

$$\langle f, g\rangle = \sum_{i=0}^{N-1} \int_{u_i(I)} f(x) g(x)\, dx$$
$$= \frac{1}{N} \sum_{i=0}^{N-1} \int_I (sf(x) + \lambda_i(x))(sg(x) + \mu_i(x))\, dx.$$

Solving for $\langle f, g\rangle$ gives

$$\langle f, g\rangle = \frac{1}{N(1-s^2)} \sum_{i=0}^{N-1} (\langle \lambda_i, \mu_i\rangle + s(\langle f, \mu_i\rangle + \langle g, \lambda_\lambda\rangle)). \quad \text{(A.1)}$$

Similarly, for any nonnegative integer n we have

$$\langle f, x^n\rangle = \frac{1}{N^{n+1} - Ns} \sum_{i=0}^{N-1} \left(\langle \lambda_i, (x+i)^n\rangle + s \sum_{j=0}^{n-1} \binom{n}{j} i^{n-j} \langle f, x^j\rangle \right). \quad \text{(A.2)}$$

Observe that (A.2) can be used recursively to calculate $\langle f, x^n\rangle$. Thus if λ_i and μ_i are polynomials then (A.1) provides an explicit formula for $\langle f, g\rangle$.

Note added in proof. After the completion of this paper, the authors learned of work by T. N. T. Goodman *et al.* (Wavelets in wandering subspaces, *Trans. Amer. Math. Soc.*, to appear) and Ch. Micchelli (Using the refinement equation for the construction of pre-wavelets VI: Shift invariant subspaces, *in* "Approximation Theory, Spline Functions, and Applications" (S. P. Singh, Ed.), NATO ASI Series C, Vol. 356, pp. 213–222) which overlaps with some of the results in Section 3.

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